

Global Artificial Boundary Conditions for Computation of External Flows with Jets

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We propose new global artificial boundary conditions (ABCs) for computation of flows with propulsive jets. The algorithm is based on application of the difference potentials method (DPM). Previously, similar boundary conditions have been implemented for calculation of external compressible viscous flows around finite bodies. The proposed modification substantially extends the applicability range of the DPM-based algorithm. We present the general formulation of the problem, describe our numerical methodology, and discuss the corresponding computational results. The particular configuration that we analyze is a slender three-dimensional body with boat-tail geometry and supersonic jet exhaust in a subsonic external flow under zero angle of attack. Similar to the results obtained earlier for the flows around airfoils and wings, current results for the jet flow case corroborate the superiority of the DPM-based ABCs over standard local methodologies from the standpoints of accuracy, overall numerical performance, and robustness.

Introduction

MANY typical problems in aerodynamics, including those that present immediate practical interest, for example, flows around aircraft, are formulated on infinite domains. It is, however, obvious that any discretization used for solving such problems on the computer must be finite. Therefore, any numerical solution methodology for these problems has to be supplemented (or rather preceded) by a special technique that helps create such finite discretizations.

A widely used approach to this problem is based on truncating the original flow domain prior to the actual discretization and numerical solution. Subsequently, one can construct a finite discretization on the new bounded computational domain using one of the standard techniques: finite differences, finite elements, or other. However, both the continuous problem on the truncated domain and its discrete counterpart will be subdefinite unless supplemented by the appropriate closing procedure at the external computational boundary. This is done by using artificial boundary conditions (ABCs); the word artificial emphasizing here that these boundary conditions are necessitated by numerics and do not come from the original physical formulation.

The ideal or, in other words, exact, ABCs are obviously those that would drive the error associated with domain truncation to zero. However, numerically efficient procedures of this kind cannot be attained routinely except in model (mostly one-dimensional) problems, and therefore, for typical applications one uses primarily different approximate rather than exact methodologies.

The nature of the difficulties associated with constructing the exact ABCs is that in most cases such boundary conditions appear nonlocal (in space and also in time for unsteady problems). Although the corresponding computational algorithms are robust and highly accurate, they can be cumbersome and typically apply only to rather simple geometries. On the other hand, the alternative local approaches that yield inexpensive and geometrically universal numerical procedures may often lack accuracy in computations, which, in turn, necessitates choosing excessively large computational domains. Basically, higher accuracy due to boundary conditions implies that more of the nonlocal nature of exact ABCs has to be taken into consideration. As a consequence, to avoid extra complexity due to the nonlocality of boundary conditions, most of the modern production algorithms in computational fluid dynamics (CFD) still employ local ABCs that are based on simplified flow models. The possibility to use local ABCs comes, as mentioned, at the expense of running the cases on large domains.

Generally, it has been demonstrated theoretically and computationally in both CFD and other areas of scientific computing that the treatment of ABCs may have a profound impact on the overall performance of numerical algorithms and interpretation of the results. The literature on various ABCs' techniques is extensive; a detailed review may be found in work by Givoli,^{1,2} as well as in a more recent paper by Tsynkov.³

The construction of ABCs based on the difference potentials method (DPM) by Ryaben'kii⁴⁻⁶ was an attempt to combine in one technique the advantages relevant to both local and global methodologies.⁷⁻¹⁷ These boundary conditions employ finite difference counterparts to Calderon's pseudodifferential boundary projection operators and generalized potentials that have been first proposed in work by Calderon¹⁸ and then also studied by Seeley.¹⁹ The DPM-based ABCs have been successfully implemented along with NASA-developed multigrid Navier-Stokes solvers for the calculation of two-dimensional (solver FLOMG by Swanson and Turkel²⁰⁻²²) and three-dimensional (solver TLNS3D by Vatsa and Wedan²³ and Vatsa et al.²⁴) compressible viscous flows around airfoils (NACA0012, RAE2822) and wings (ONERA M6).

In many numerical tests the DPM-based boundary conditions have consistently outperformed the standard local methods from the standpoints of accuracy, multigrid convergence rate, and overall robustness (they allow for a substantial reduction of the domain size while preserving the accuracy and may also speed up the convergence of multigrid iterations by up to a factor of three; that is, they would require only about one-third of the original number of multigrid cycles for reducing the initial residual by a prescribed factor). Note, the standard local boundary conditions for external flows that are referred to are typically based on one-dimensional characteristics analysis for the front or inflow part of the artificial boundary

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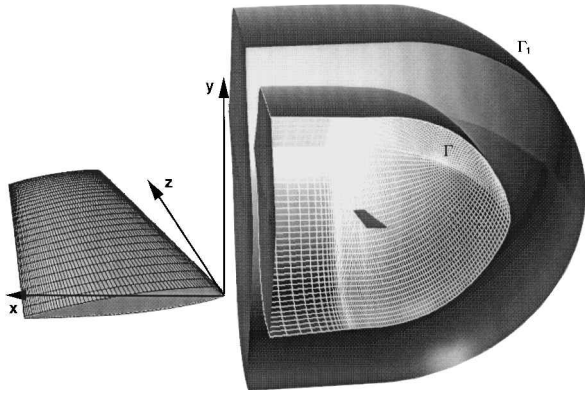


Fig. 1 Schematic geometric setup for the ONERA M6 wing; wing on the left is enlarged.

and specification of the freestream pressure and extrapolation of all other quantities on the rear or downstream portion of the outer boundary. This treatment may or may not be supplemented by the point-vortex correction²⁵ for the two-dimensional case. An example of geometry in three dimensions is shown in Fig. 1.

All of the problems analyzed previously in the DPM framework (see the aforementioned references) can actually be characterized as pure external flows. In this paper, we for the first time incorporate a new and essentially different physical element into the formulation of the problem; namely, we will consider external flows around configurations with jet exhaust. The problems of this kind have never been studied by means of the DPM before, and including this new flow phenomena into the range of admissible formulations for the DPM-based methodology substantially enlarges the scope of its capabilities. Moreover, as different flows with jets are frequently encountered in aerospace applications, the possibility of computing external aerodynamics more efficiently with jet exhaust phenomena taken into account is important for both configuration analysis and design.

The material in the paper is prepared as follows. In the next section, we outline the basic DPM-based procedure as developed for pure external flows; in the section that follows we describe the changes that are necessary for incorporating the jet exhaust flows. Then, we present the Numerical Results and Conclusions.

DPM-Based ABCs: Basic Algorithm

In this section, we outline the corresponding derivation from Ref. 15. The paper by Tsynkov¹⁶ contains a substantially more detailed account on how to set the three-dimensional DPM-based ABCs.

We consider a steady-state flow of a viscous, compressible, perfect gas past a finite three-dimensional configuration. The flow is uniform and subsonic at infinity; it is also symmetric with respect to the Cartesian plane $z = 0$. The hydrodynamic equations are discretized and integrated on a grid generated around the immersed body(ies). The grid actually defines a bounded computational domain; the ABCs that would close the truncated problem should be set at the external coordinate surface of the grid. Let us denote this surface Γ ; for a one-block curvilinear C-O-type boundary-fitted grid around the ONERA M6 wing, the schematic geometric setup is shown in Fig. 1.

The outermost coordinate surface of the grid is designated Γ_1 (Fig. 1); it represents the ghost nodes (or ghost cells for the finite volume formulation). Clearly, when the stencil of the scheme used inside the computational domain is applied to any node from Γ , it generally requires some ghost cell data. Unless the required data are provided, the finite difference system solved inside the computational domain appears subdefinite, that is, it has fewer equations than unknowns. Therefore, in a practical framework, the closure of the discretized truncated problem means specification of the solution values at the ghost cells. This will be done with the DPM-based ABCs so that the boundary data used for the closure admit an exterior complement that solves the problem outside the computational domain. As soon as the data in the ghost cells have been obtained

as functions of the data in the interior cells (Γ_1 as a function of Γ), the corresponding relations can be incorporated into the actual solver used inside the computational domain. If, for example, this is an iterative solver (very often the case), then one has to update the ghost cells at each iteration to advance to the next time step.

To construct the boundary conditions, we first assume that the flow perturbations against constant freestream background are small in the far field and consider the linearized problem outside the computational domain, that is, outside Γ . It is important to emphasize that the possibility of far-field linearization, that is, the possibility to retain only the first-order terms with respect to perturbations in the governing equations, requires special justification, in particular, when analyzing transonic flows. We do not present the corresponding argument here; a simple asymptotic analysis in the framework of the full potential model that justifies the far-field linearization in three dimensions may be found in previous work.^{15,16} Of course, even if we know that the far field is linear, we still cannot say a priori whether the linearization outside Γ is possible for a particular configuration of the domains. Clearly, for a very large computational domain, one can linearize the flow outside Γ , and as we approach the source of perturbations (the immersed configuration), the validity of linearization is verified a posteriori (e.g., see Refs. 8, 9, 12, 15, and 16).

We will be considering the entire problem in the framework of the thin-layer equations (as opposed to the full Navier-Stokes equations). This simplified flow model still retains all of the essential properties pertinent to the class of problems that we are studying, and at the same time it is less expensive numerically. In the far field, the thin-layer equations are linearized against the constant freestream background and supplemented by the condition of vanishing of all flow perturbation at infinity, which corresponds to the freestream limit of the solution. The actual form of the linearized governing equations in the far field is presented in Refs. 15 and 16.

We discretize the linearized thin-layer system with the second order of accuracy on an auxiliary Cartesian grid; a detailed description of the resulting finite difference scheme may be found in Ref. 16. The DPM will provide us with the complete boundary classification of all of those and only those exterior grid vector functions that solve this discretized system outside the computational domain and meet the boundary condition at infinity (vanishing of all perturbations). The foregoing boundary classification will be obtained as an image of a special projection operator, which can be considered a discrete analog to Calderon's pseudodifferential boundary projection.^{18,19} The projection operators act on the grid functions defined as boundary traces of the solution. In actual computations, the boundary conditions are set as follows. Every time we need to update the ghost cells, we take an appropriate set of data from inside Γ (subsequently discussed), project it onto the subspace in the entire space of boundary data that admits the correct exterior complement, and obtain the ghost cell values by calculating this complement on Γ_1 .

The implementation of the DPM-based ABCs starts with splitting the nodes of the auxiliary Cartesian grid into two distinct groups: those that are inside Γ and those that are outside Γ . Applying the stencil of the scheme that we use to each node of both groups, we consider the intersection of the grid sets swept by the stencil. This intersection is called the grid boundary γ ; it is a multilayered fringe of nodes of the auxiliary Cartesian grid located near and straddling the continuous boundary Γ .

For any function u on the Cartesian grid, we define its trace $Tr_\gamma u$ on γ as merely a restriction. For any grid function u_γ specified on γ , we introduce the generalized potential Pu_γ with the density u_γ ; the generalized potential is defined on the auxiliary Cartesian grid on the boundary γ and outside it. The generalized potential is obtained as a solution of the special auxiliary problem (AP) driven by the right-hand side that depends on u_γ ; the formal construction of this right-hand side is the same in two- and three-dimensional cases (see Refs. 7, 11, 12, and 16 for details). The AP is formulated on a finite auxiliary domain, which is a Cartesian parallelepiped that fully contains Γ_1 . The formulation of the AP involves periodization in the cross-stream and spanwise directions y and z , respectively. The periods are chosen sufficiently large to guarantee that the periodic solution considered on a finite fixed neighborhood of Γ and Γ_1

provide a good approximation to the theoretical nonperiodic solution. The latter can be thought of as a limit when the periods in y and z approach infinity. The approximation of a nonperiodic solution by the periodic one on a fixed subinterval as the period increases is discussed elsewhere.^{7,11,12} The boundary conditions for the AP in the streamwise direction x are constructed semi-analytically using explicit mode selection that would guarantee the desired far-field behavior of the solution (see Refs. 15 and 16).

The AP allows us to calculate the generalized difference potential $\mathbf{P}u_\gamma$ for any grid density u_γ specified on γ . In the vicinity of the computational domain, the potential approximates the solution to the linearized equations subject to the zero boundary condition at infinity (the accuracy of this approximation can be controlled by choosing the periods in y and z). The composition of the operators \mathbf{Tr}_γ and \mathbf{P} , $\mathbf{P}_\gamma \equiv \mathbf{Tr}_\gamma \mathbf{P}$, is a projection, $\mathbf{P}_\gamma^2 = \mathbf{P}_\gamma$, and it is a discrete counterpart of Calderon's boundary projection^{18,19} for the linearized thin-layer system. The image of this projection, $\text{Im } \mathbf{P}_\gamma$, contains all of those and only those u_γ that are traces of the exterior difference solutions to the linearized thin-layer equations that satisfy the boundary conditions of the AP. In the sense of the foregoing approximation (periodization), $\text{Im } \mathbf{P}_\gamma$ contains all of those and only those u_γ that are traces of the exterior difference solutions to the linearized thin-layer equations that satisfy the zero boundary conditions at infinity.

Having constructed the procedure for calculating the potentials and projections for the discrete linearized thin-layer equations, we can now close the system inside the computational domain, that is, obtain the ABCs. First, we take \mathbf{u} and $\partial \mathbf{u} / \partial n$ on Γ , n is the normal (these data are available from inside the computational domain), and using interpolation \mathbf{R}_Γ along Γ and the first two terms of Taylor's expansion (denoted π_γ) obtain u_γ :

$$u_\gamma = \pi_\gamma \mathbf{R}_\Gamma \left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial n} \right) \Big|_\Gamma \quad (1)$$

We need to calculate the potential $\mathbf{P}v_\gamma$ for the density $v_\gamma = \mathbf{P}_\gamma u_\gamma$ and interpolate it to the nodes Γ_1 :

$$u|_{\Gamma_1} = \mathbf{R}_{\Gamma_1} \mathbf{P}v_\gamma \equiv \mathbf{R}_{\Gamma_1} \mathbf{P}u_\gamma \quad (2)$$

Finally, the ABCs are obtained in the operator form

$$u|_{\Gamma_1} = \mathbf{T} \left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial n} \right) \Big|_\Gamma \quad (3)$$

where \mathbf{T} is composed of the operations of Eqs. (1) and (2). Boundary condition (3) is applied every time we need to update the ghost cells when solving the interior problem, for example, on every iteration. The implementation of ABCs (3) can either be direct or involve preliminary calculation of the matrix \mathbf{T} . In the latter case, the runtime implementation of the ABCs (3) is reduced to a matrix-vector multiplication. Numerical results for flows around the ONERA M6 wing obtained with the DPM-based boundary conditions (3) are summarized in work by Tsynkov and Vatsa¹⁵ and Tsynkov.^{16,17}

Application to Jet Flows

The major difference between the formulation of the preceding section and the flow with jet exhaust is that in the vicinity of the jet we can no longer claim that flow perturbations against the freestream background are small. Indeed, inside the propulsive jet, the speed of the flow is typically much higher than the one in the surrounding area; moreover, other parameters, for example, temperature, may also differ substantially. Therefore, the linearization of the flow against a constant freestream background everywhere is, generally speaking, not valid in this case.

However, in many applications (in particular, aerospace) one can clearly distinguish between those parts of the overall flow that contain jet(s) and the remaining areas. Therefore, the most comprehensive way to develop the far-field linearization in this situation will apparently be to use the appropriate asymptotic solutions for jets (e.g., see Abramovich²⁶) in the corresponding regions as a background. For flow regions outside the jet, it is always reasonable to

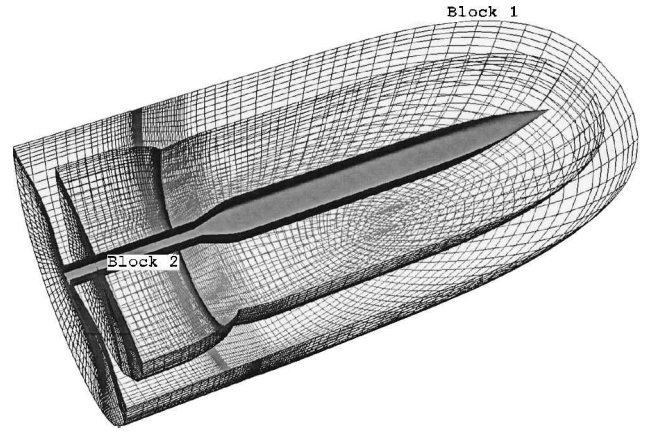


Fig. 2 Schematic geometric setup for the slender body with jet exhaust.

assume that the foregoing linearization around a uniform freestream solution will still be valid there.

The particular setting that we will be studying hereafter is schematically shown in Fig. 2. (The meaning of the two external grid surfaces is the same as Γ and Γ_1 in Fig. 1.) It includes a three-dimensional slender body (symmetric with respect to the $z = 0$ plane but not axially symmetric, i.e., not a body of revolution) with sharp nose and boat-tail aft configuration. The rearmost plane surface of the body (not shown explicitly in Fig. 2) is actually a location of the nozzle outlet; the outlet is rectangular in cross section. The exterior flow is subsonic with the freestream Mach number $M_0 = 0.6$ and zero angle of attack; the jet that is discharged from the outlet is supersonic, $M_j = 1.6$, and confluent with the exterior flow.

The specific shape of the body (Fig. 2) as well as parameters of the flow have been previously proposed for numerical study and analyzed in work by Compton.²⁷ In this original work,²⁷ Compton had calculated external flow with the propulsive jet and also considered the interior portion of the flow, namely, the flow in the actual nozzle located inside the afterbody (this nozzle flow obviously produces the jet). For our study, we have generated new grids and also simplified the overall formulation by eliminating the nozzle and specifying instead the uniform supersonic flow conditions at the nozzle outlet, that is, at the place where the jet starts. Compton's goal²⁷ was to assess the performance of different turbulence models including their prediction capabilities for the flow inside the nozzle; our goal is to assess the performance of different external boundary conditions for the flows with jet exhaust. We, therefore, think that the aforementioned simplification is justified.

Our typical grid consists of two blocks: block 1 of C-O type is for the exterior flow and block 2 of H-O type is for the jet portion (Fig. 2). Of course, this subdivision can only have an approximate meaning because the jet will obviously tend to spread while propagating downstream; basically, it means that the shear layer between the jet and coflow is located in the vicinity of the block interface. On this interface, the two grid blocks are point matched, which is a requirement for TLNS3D.

As mentioned, the exterior flow is subsonic, and the jet is supersonic (other parameters of the flow will be pointed out later). The standard boundary conditions in TLNS3D for this two-block jet flow case include one-dimensional characteristics for external inflow (block 1, upstream portion of the boundary), specification of the freestream pressure with extrapolation of all other quantities for external outflow (block 1, downstream portion of the boundary), extrapolation of all quantities for the jet downstream boundary (block 2), and specification of all quantities for the jet inflow boundary (block 2). The boundary conditions on the solid surface of the body are standard no-slip conditions. Extrapolation of all flow quantities at the jet outflow boundary is justified because, as shown by numerous simulations, the core of the jet remains supersonic even at large distances downstream of the body, at least as far as 40–50 nozzle calibers away.

The primary goal of this paper is to develop an alternative to the foregoing local boundary conditions for the jet flow case, global

ABCs similar to those described in the preceding section, and to compare the performance of the two techniques. A direct implementation of the ABCs (3) will, however, encounter a major obstacle in this case: As has been mentioned, we cannot linearize against the freestream background in the jet region and, therefore, cannot directly implement the operator \mathbf{T} of ABCs (3) over the entire external boundary because this operator is obtained on the basis of the thin-layer system linearized against a uniform freestream background. Of course, if we linearized the flow against a constant freestream background outside the jet and against some approximate asymptotic solution in the jet region²⁶ and then used the corresponding linear system (it will have variable coefficients) to construct the operator analogous to \mathbf{T} of Eq. (3), then we could have applied the boundary conditions (3) straightforwardly as done in previous work^{15,16} for flows with no jets. Computation of the new operator \mathbf{T} in this framework will, in turn, require a different construction of the AP, certainly more elaborate (because of variable coefficients) and possibly more expensive than the one described in the preceding section. One way of largely eliminating the difficulties associated with variable coefficients is apparently to take advantage of the supersonic nature of the jet and use marching-type algorithms in a subdomain of the new AP domain. Although this may make the whole foregoing program feasible, we consider its implementation as future work. In this paper, we present the algorithm based on boundary conditions (3) with minimal alterations.

As the ABCs (3) obviously cannot be applied in the jet area, that is, on that portion of the artificial boundary where the jet exits the domain, we need another procedure. The most natural choice will be to extrapolate all flow quantities downstream at the outflow boundary because the core of the jet remains supersonic even at large distances away from the nozzle outlet. Of course, we cannot actually predict where on the downstream boundary the flow actually becomes subsonic, that is, where the extrapolation ceases to be applicable. However, we have observed that for the particular case under study we can extrapolate at least on the entire downstream boundary of the second grid block (Fig. 2). Thus, extrapolation of all flow quantities will be used henceforth as downstream boundary conditions for block 2.

In the standard procedure, the downstream boundary conditions for grid block 1, that is, on the rest of the outflow boundary, are based on the specification of freestream pressure and extrapolation of all other quantities. Basically, these boundary conditions are relevant for subsonic outflow. In practice, some portion of the downstream boundary of block 1 may also be supersonic; in this case, however, the implementation of these pressure boundary conditions does not lead to noticeable errors because the streamwise variations of pressure away from the nozzle are small [the jet is close to design and is slightly overexpanded (see subsequent text)] and therefore, specification of the freestream pressure and extrapolation from the interior are both procedures with acceptable accuracy.

To replace local boundary conditions on the outer boundary of block 1 (the region outside the jet) by the more accurate global ABCs, we use relation (3). However, in formula (3) both the input and output are global, that is, the operator \mathbf{T} not only provides the ghost cell data along the entire boundary but also requires the data along the entire (penultimate) boundary as driving terms. By using extrapolation downstream in the jet core instead of using Eq. (3), we make sure that the possibly erroneous data from the global procedure are not used on this part of the boundary. However, because the global operator \mathbf{T} is constructed on the basis of the linearization against a constant background, which is not valid in the jet area, plugging the actual flow quantities (including the jet profile) into the right-hand side of Eq. (3) may potentially generate errors along the entire outer boundary.

On the other hand, it has been verified for model examples¹⁴ and also seen for more complex cases that, typically, closely located boundary nodes more strongly influence one another than the remote ones. This is a reasonable behavior from the standpoint of physics; in the structure of operators \mathbf{T} reflects this behavior so that, although the matrix is dense (nonlocality), its near-diagonal terms are much larger than the off-diagonal ones (for systems as opposed to scalar equations, it will be a similar blockwise structure). The

specific rate of decay for the off-diagonal terms can probably be obtained only for the most elementary formulations, for example, the Laplace equation with periodic boundary conditions. However, although we cannot obtain analytical estimates for the kernels involved in operators \mathbf{T} of Eq. (3), we can still make use of the actual (blockwise) off-diagonal decay in the numerical experiments. In practical terms, this implies that although substituting the jet profile into the right-hand side of Eq. (3) violates the small perturbations assumption, the associated error on the left-hand side of Eq. (3) will mostly be concentrated again in the jet area, where the results are not used for boundary conditions anyway because they are overridden by extrapolation.

Thus, the actual combined DPM-based ABCs that we employ for computation of the foregoing jet flow case are the following. For most of the outer boundary (except the near-jet area) we use formula (3) while substituting the actual flow profile in its right-hand side. For the jet core (outflow boundary for grid block 2), we extrapolate all flow quantities downstream. For the small intermediate portion of the downstream boundary (near the jet core) we extrapolate all quantities except pressure; the latter is prescribed from its free-stream value. In fact, we have observed that within a certain range (5–30 cells of the fine grid described in Numerical Results), the actual location of where to switch from the pressure boundary condition to formula (3) does not exert much influence on either the final accuracy or multigrid convergence rate. In the next subsection, we provide an additional justification for the use of this procedure.

Jet Outflow Boundary Conditions

To describe and explain the specific boundary treatment in the vicinity of the jet exit through the boundary, we start by considering the following model problem, disregarding for a moment the connection to the global boundary procedure already described.

A model problem describing the error due to inaccurate outflow pressure data for the steady Euler equations is

$$\begin{aligned} Ae_x + Be_y + Ce_z &= 0, & x < 0 \\ p &= g(y, z), & x = 0 \\ -\infty < y, z &< \infty \end{aligned} \quad (4)$$

where $e = (\rho, u, v, w, p)^T$ denotes the error and A , B , and C are constant matrices. We assume that the boundary data have compact support outside a small portion of the boundary; that is,

$$g(y, z) = 0, \quad |y, z| \geq \delta \quad (5)$$

We also assume that the base flow is subsonic and moves to the right. The problem (4) and (5) is a model for the error in an approximate solution with correct outflow boundary data given on $|y, z| \geq \delta$ and erroneous data on $|y, z| < \delta$.

The relation of the model problem (4) and (5) to the specific outflow problem in this paper can briefly be described as follows. The global boundary procedure far away from the jet and the extrapolation procedure^{28,29} in the supersonic part of the jet lead to very small errors, that is, $|g| \approx 0$. In an intermediate domain between the supersonic part of the jet and the part where the global boundary conditions are used, pressure with erroneous data is specified, that is, $|g| \approx \mathcal{O}(1)$ in that part of the domain.

Note that for problems with boundary conditions in the x (or streamwise) direction, it makes little difference if one considers the inviscid Euler equations instead of the viscous thin-layer Navier–Stokes equations because the number and nature of the boundary conditions required in the x direction are the same for the two sets of equations.

Let $\Omega_L^n = [-(L+1), -L] \times \mathbb{R}^{n-1}$, where n is the number of spatial dimensions (see Fig. 3). The following theorem describes the error distribution in the half-space $x < 0$.

Theorem 1: The error e in Eqs. (4) and (5) satisfies

$$\|e\|_{L_2(\Omega_L^n)} \leq C \sqrt{\frac{1 - \bar{u}^2}{1 - Q^2}} |\Delta p_{\max}| \left[\frac{\delta}{\sqrt{L}} \right]^{n-1} \quad (6)$$

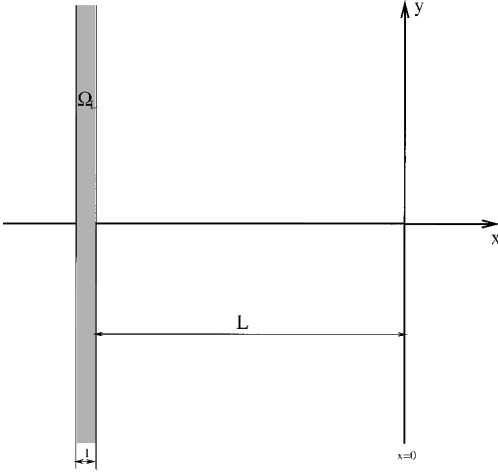


Fig. 3 Domain Ω_L^2 , where L is the distance from Ω_L^2 to the boundary, $L \gg 1$.

where C is a constant and $|\Delta p_{\max}|$ is the maximal magnitude of the pressure error in $|y, z| < \delta$, $\bar{Q}^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2$ for $n=3$ and $\bar{Q}^2 = \bar{u}^2 + \bar{v}^2$ for $n=2$.

Theorem 1 means that by measuring the error in a local L_2 norm on the fixed domain Ω_L^2 , an error decay can be observed. Note that if the error was measured by computing the L_2 norm in the whole computational domain, no error decay could be observed. The proof of Theorem 1 involves a straightforward application of the theory of Ref. 30. Numerical experiments that verify the decay rate (6) can be found in Ref. 31.

For our specific outflow problem with erroneous data given on the intermediate domain between the supersonic part of the jet and the part where global boundary conditions are used, Theorem 1 means that the error decays with the rate δ^2/L away from the outflow boundary. Furthermore, in our specific flow problem, we have a slightly overexpanded jet, which means that the maximum pressure error $|\Delta p_{\max}|$ in Eq. (6) is rather small.

Effective Reynolds Number

To calculate the operator T of Eq. (3), we are solving the AP for the linearized thin-layer equations. These equations formally involve the molecular Reynolds number of the flow. However, because the actual flows that we are studying are turbulent, to integrate the thin-layer equations numerically one complements them with turbulence models inside the computational domain. These models may be complex and require solving some additional differential equations (see next section).

For the simplified linearized far-field representation, we do not use these accurate and sophisticated turbulence models. However, we still need to account for the corresponding turbulent mixing and dissipation, at least in an approximate way. In previous work,⁹ Tsynkov et al. have used the concept of effective turbulent viscosity for the far field and calculated the effective turbulent Reynolds number using the fact that the laminar and turbulent plane wakes behind the body have the same asymptotic behavior.³²

The asymptotic behavior of laminar and turbulent circular jets is also known to be the same.^{26,32} It involves a linear increase in width and a decrease in centerline velocity inversely proportional to the distance from the source. The virtual kinematic viscosity (incompressible case) can be considered constant through the entire jet region. Although we do not use boundary conditions (3) in the core of the jet, the outer portions of the shear layer region are still covered by the global procedure; therefore, we need to provide the effective value of $1/Re$ for the linearized thin-layer equations.

The jet that we are studying is rectangular in its initial cross section (see next section for particular details); however, its shape will approach circular farther away from the outlet. Therefore, we will use the results obtained for circular jets to find an approximate value for the effective Reynolds number. First, we notice that the universal velocity profiles in a cross section of an incompressible submerged jet, that is, the jet that propagates through a medium at rest, are

the same as those obtained for the excess velocity of the jet propagating in a coflow.²⁶ Moreover, many experimental observations corroborate²⁶ that the same universal profiles remain valid for a compressible supersonic jet spreading through either a stationary or moving medium. Of course, whereas the profiles are universal, the actual spreading rate for the jet will differ for different cases. Second, for the particular case under study (the ratio of stagnation temperatures is $T_j^*/T_0^* = 0.936$, the design pressure ratio is $p_j^*/p_0|_{\text{design}} = 4.25$ at $M_j = 1.6$, whereas the actual pressure ratio is $p_j^*/p_0 = 4.00$, and the jet is slightly overexpanded), the initial value of the compressibility parameter²⁶ is $\bar{\rho} = \rho_0/\rho_j = 1.41$ and the initial velocity ratio is $m = u_0/u_j = \sqrt{(T_0/T_j)M_0/M_j} = 0.459$. According to Ref. 26, these values are within the range ($0 \leq m \leq 0.6$ and $0.3 \leq \bar{\rho} \leq 1.43$) for which the correction due to compressibility for the spreading rate b of the jet can be taken into account by calculating it as

$$b_{\text{comp}} = cx[(1 + \bar{\rho})/2][(1 - m)/(1 + \bar{\rho}m)] \quad (7a)$$

instead of the old expression

$$b_{\text{inc}} = cx[(1 - m)/(1 + m)] \quad (7b)$$

which is relevant for the incompressible flow; c in formulas (7) is a constant, and x is the distance from the source.

According to the measurements referenced by Schlichting,³² for a submerged incompressible jet $b_{1/2} = 0.0848x$, where $b_{1/2}$ is half-width of the jet at half-depth. Substituting this into the solution for laminar jet^{26,32}

$$b_{1/2}/x = 5.27(v/\sqrt{K})$$

one obtains the virtual kinematic viscosity equation³²:

$$\nu_T = 0.0161\sqrt{K} \quad (8)$$

where K is the total kinematic momentum flux. Because the velocity profiles are universal, for the jet with coflow we only need to multiply the spreading rate by $(1 - m)/(1 + m)$ according to formula (7b), and for the compressibility correction we use Eq. (7a), which altogether yields

$$\nu_T = 0.00636\sqrt{K} \quad (9)$$

As mentioned, the boundary condition that we specify for the jet inflow is a uniform supersonic profile across the entire nozzle outlet. Therefore, the quantity K can be obtained by multiplying the square of the excess velocity (relative velocity of the jet with respect to the velocity of coflow) by the area of the outlet σ , that is, $K = (u_j - u_0)^2 \sigma$. Then, the effective turbulent Reynolds number is calculated as $Re_T = UL/\nu_T$, where U is the characteristic speed and L is the characteristic length. For the particular setting under study, it is reasonable to assume that $U = |u_j - u_0|$ and $L = \sqrt{\sigma}$. Consequently, from Eq. (9) we conclude that

$$Re_T = 0.00636^{-1} \approx 157 \quad (10)$$

In our computations, the actual value of Reynolds number for the linearized thin-layer system was taken from Eq. (10).

Numerical Results

The particular geometry of the body shown in Fig. 2 is the following: rectangular cross section $y \times z = 6.2 \times 6.8$ with rounded edges, sharp nose, and boat-tail afterbody; total length in the x direction is 63; and rectangular nozzle outlet $y \times z = 2.62 \times 5.12$. A full description may be found in the work by Compton.²⁷

The geometry and the flow are symmetric with respect to the plane $z = 0$ (zero angle of attack). For our computations we have used three different domains with successively reduced dimensions (see Fig. 4); domain I (or large domain) with the diameter of about 30 calibers of the body was used for calculating the reference solutions, domain II is 0.36 or about one-third of the size of domain I in each direction, and domain III is 0.22 or about one-fifth of the size of domain I in each direction.

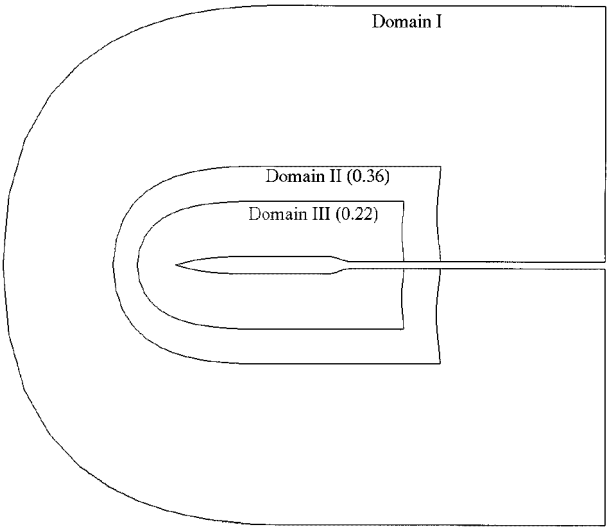


Fig. 4 Three computational domains for the jet flow: projection onto the $z = 0$ plane.

As mentioned, to integrate the thin-layer equations on the curvilinear grid shown on Fig. 2 we use the code TLNS3D by Vatsa and Wedan²³ and Vatsa et al.²⁴ This is a central-difference code with five-stage explicit pseudotime Runge–Kutta relaxation used for obtaining steady-state solutions. The code employs a local Courant step, semi-implicit residual smoothing, and multigrid for accelerating the convergence. In our computations, we used either three or two nested grid levels with V cycles (depending on the grid dimension); this multilevel V-cycle algorithm is, in fact, a final stage of the full multigrid (FMG) procedure. In addition, to improve the convergence to steady state, the solver is preconditioned according to the methodology of Ref. 33.

The DPM-based ABCs are implemented only on the finest grid for the V cycle in the final FMG stage; the boundary data for coarser levels are provided by the coarsening procedure. Moreover, even on this finest grid we implement the DPM-based ABCs only on the first and the last Runge–Kutta stages, which has been found^{15,16} to make very little difference compared to the implementation on all five stages. The boundary data for the three intermediate stages are provided from the DPM-based ABCs on the first stage.

To account for the turbulent phenomena, the solver is also supplemented with Menter’s two-equation turbulence model.³⁴ The actual molecular Reynolds number based on unit length is $Re = 3.21 \times 10^5$, Prandtl number is $Pr = 0.72$, and specific ratio is $\kappa = 1.4$.

We have used several different grids to calculate the jet flow; in all cases we kept the normal spacing near the solid surface the same: $\approx 3 \times 10^{-4}$. All grids are stretched, and the cell size increases away of the body in geometric progression. The dimension of the C–O grid block 1 for domain I was $i \times j \times k = 385 \times 77 \times 33$ (i is the streamwise C-type coordinate, j is the radial coordinate, and k is the circumferential cross-stream O-type coordinate, quarter-circle). The dimension of the H–O grid block 2 for domain I was $i \times j \times k = 81 \times 77 \times 65$ (i is streamwise, j is radial, and k covers half-circle). We will further refer to this grid as fine. On the fine grid, we have calculated two reference solutions, one with standard ABCs and another with global ABCs. Because the artificial boundary for domain I is located sufficiently far away from the body, the difference between the corresponding results is negligible. In Figs. 5 and 6 we show convergence histories for this case: The residual of the continuity equation is plotted vs work units in Fig. 5, and drag coefficient is plotted vs work units in Fig. 6. (One work unit is the cost of advancing one time step on the finest grid.)

From Figs. 5 and 6, we conclude that multigrid convergence rates are the same for local and global ABCs on domain I. Moreover, the values of total drag coefficient per unit area C_D are summarized for this case in the right column of Table 1. They differ by about one-third of 1%, which corroborates that the type of external boundary conditions has little effect on the solution itself, as well as multigrid convergence history, for large computational domains.

Table 1 Total drag coefficient per unit area				
ABCs	C_D			
	Domain III, fine grid	Domain II		Domain I, fine grid
		Coarse grid	Fine grid	
Local	—	$2.77 \pm .03$	$2.74 \pm .04$	2.506
Global	2.365	2.495	2.484	2.497

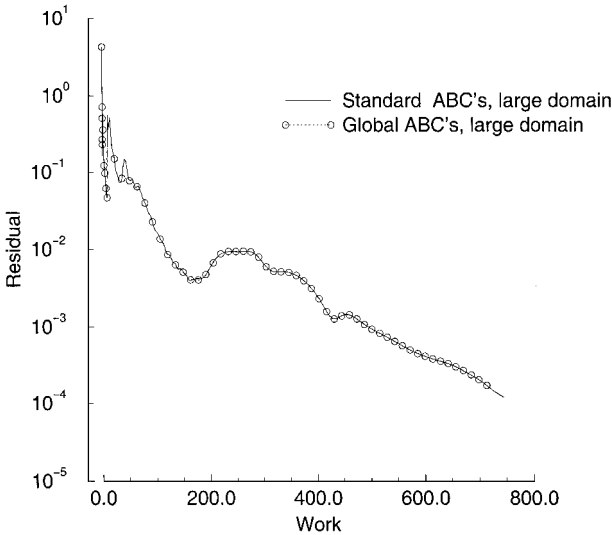


Fig. 5 Convergence histories for the residual of the continuity equation: domain I, fine grid.

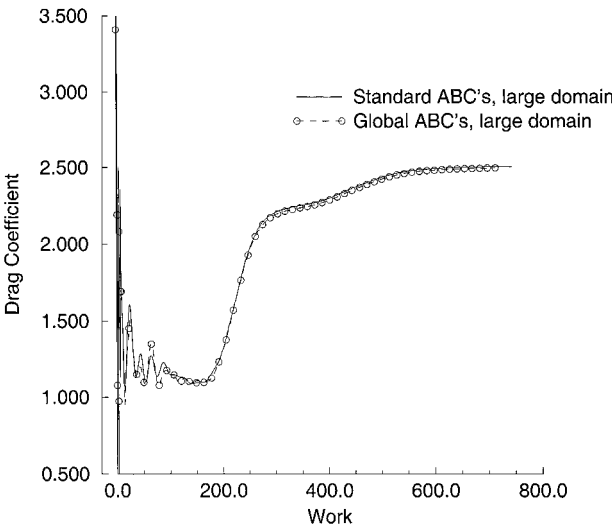


Fig. 6 Drag convergence: domain I, fine grid.

For domain II, we have computed the flow on two grids with different dimensions. The first grid has the same number of nodes as the one used in domain I; it was, in fact, constructed by scaling down the original large grid by a factor of 0.36 in each direction. We will also refer to it as fine grid. As shown in Table 1, the coefficient C_D obtained on this grid with global ABCs differs by less than 1% from its reference value, whereas the accuracy provided by local ABCs is not nearly as good, about 9% discrepancy. Moreover, because of the poor convergence (see Figs. 7 and 8), the value of C_D for local ABCs is given with the error bands indicated.

The much smaller size of domain II compared to domain I actually suggests that on domain II one can successfully compute the solution on a grid with fewer nodes. Therefore, the second grid that we have used for domain II had one-half of the original dimension in two out of three directions, block 1 $i \times j \times k = 193 \times 39 \times 33$ and block 2 $i \times j \times k = 41 \times 39 \times 65$; this grid will be referred to as coarse. Again, as follows from Table 1, global ABCs provide for an accurate

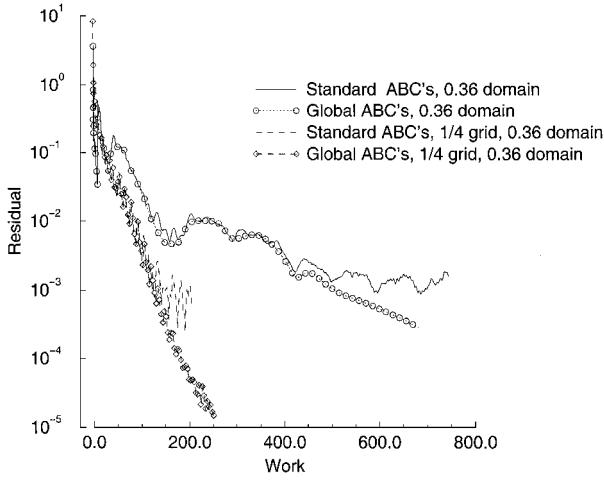


Fig. 7 Convergence histories for the residual of the continuity equation: domain II, fine and coarse grids.

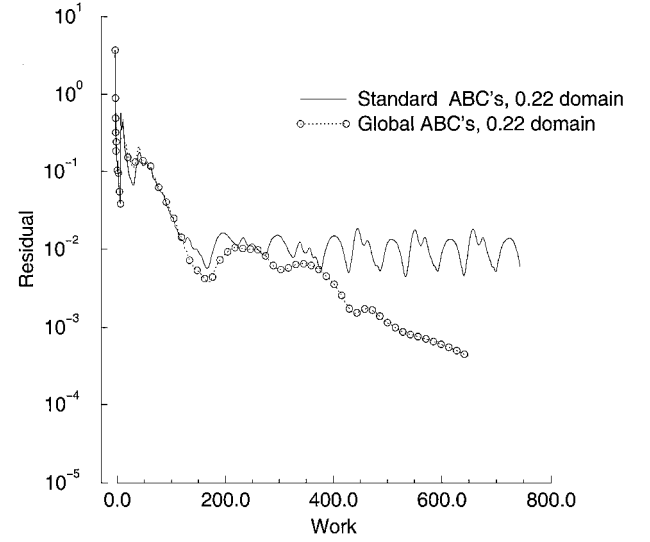


Fig. 9 Convergence histories for the residual of the continuity equation: domain III, fine grid.

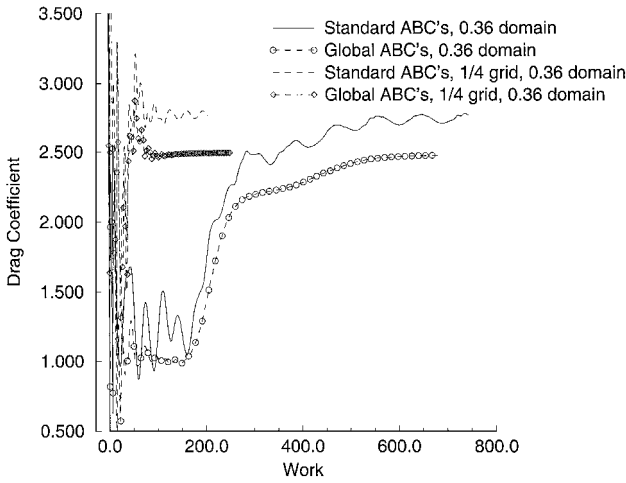


Fig. 8 Drag convergence: domain II, fine and coarse grids.

solution, whereas the accuracy of local ABCs is not sufficient and the convergence is slow (or even nonexistent). Convergence histories for domain II are presented in Figs. 7 and 8.

Because the node count for the coarse grid is only one-quarter of the node count for the fine grid, the convergence vs work for the coarse grid is about four times faster (Figs. 7 and 8), although convergence rates measured vs number of multigrid cycles will be approximately the same for both grids. Note that because of the particular grid dimensions (the issue of divisibility by 2) we have used three nested multigrid levels on the fine grid and two levels on the coarse grid. One can clearly see from Figs. 7 and 8 that the DPM-based ABCs provide for a noticeably higher multigrid convergence rate than the standard local ABCs do. Moreover, it is in fact hard to conclude from Figs. 7 and 8 whether the algorithm with local ABCs converges. If it does, the resulting C_D will be about 10% off its reference value.

On domain III, the computations were performed on the fine grid, which again was obtained by scaling down the grid from domain I (a factor of 0.22 in each direction). The algorithm with local ABCs for this domain/grid failed to converge, whereas the algorithm with global ABCs converged with the same rate as before. However, the actual computed C_D is about 5% off its reference value (Table 1). This can apparently be because the assumption of linearity (small perturbations) outside the computational domain is violated for such a small domain size. Convergence histories for domain III are presented in Figs. 9 and 10.

Computations on a coarse grid for domain III were not performed because we did not expect to recover the accurate value of C_D . However, that the algorithm with global ABCs converges on domain III corroborates the high robustness of this procedure.

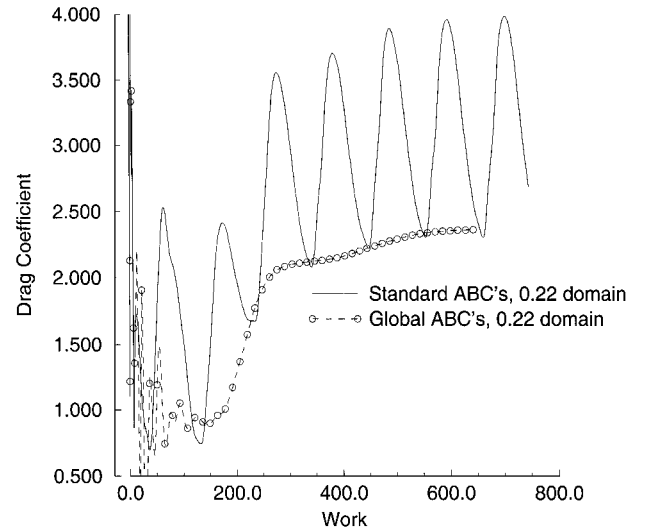


Fig. 10 Drag convergence: domain III, fine grid.

Basically, the computational results presented already allow us to see that the performance of the global DPM-based ABCs is superior to that of the standard local boundary conditions. This conclusion is supported by the faster multigrid convergence rates, as well as better accuracy of the calculated drag coefficient C_D on small domains that we obtained through the use of the DPM-based ABCs. In terms of accuracy, we corroborate this conclusion even further by presenting a more detailed account of the flow characteristics that pertain to the computed solutions (in addition to the values of the overall drag coefficient).

In Figs. 11 and 12, we present the distribution of the pressure coefficient

$$C_p = (p - p_0) / \frac{1}{2} \rho_0 u_0^2$$

where p_0 , ρ_0 , and u_0 , are static pressure, density, and absolute velocity of the flow at infinity, respectively, in the vertical plane of symmetry $z = 0$ on the afterbody portion of the analyzed configuration (Fig. 2). Figure 11 shows a longer stretch for the streamwise coordinate x normalized by the total length of the body L : $0.84 \leq x/L \leq 1.0$; Fig. 12 presents a zoomed-in view of the same distribution on a shorter stretch, $0.88 \leq x/L \leq 0.94$. Figures 11 and 12 summarize the distribution of C_p on all three computational domains on fine grids.

From Figs. 11 and 12, we see that on the large domain I the solutions obtained with global and local ABCs indeed deviate from

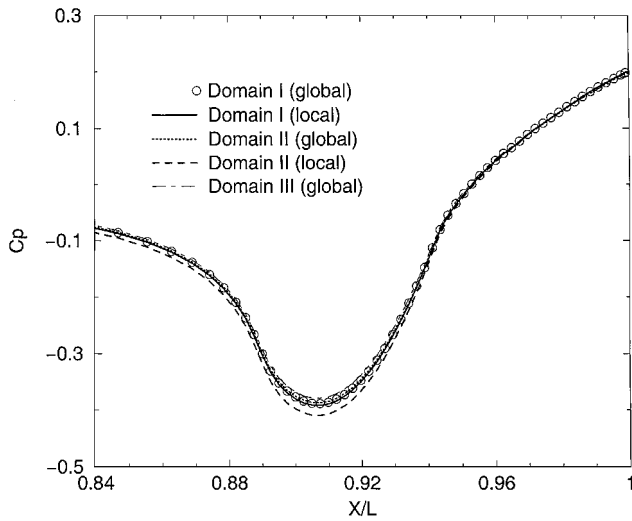


Fig. 11 Distribution of the pressure coefficient C_p on the afterbody in the symmetry plane $z = 0$.

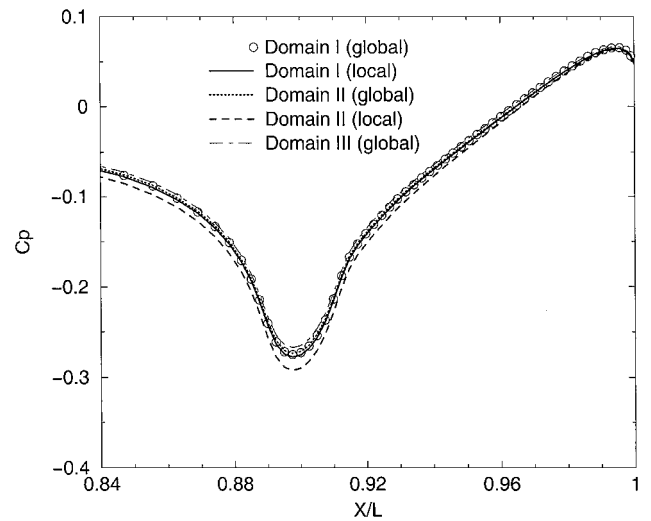


Fig. 13 Distribution of the pressure coefficient C_p on the afterbody in the symmetry plane $y = 0$.

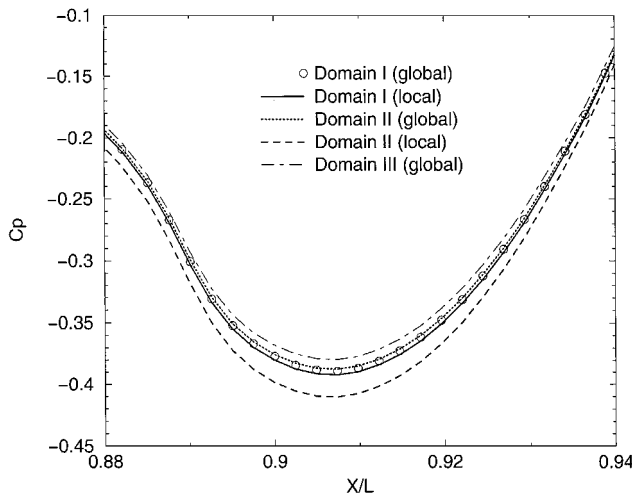


Fig. 12 Distribution of the pressure coefficient C_p on the afterbody in the symmetry plane $z = 0$: zoomed-in view.

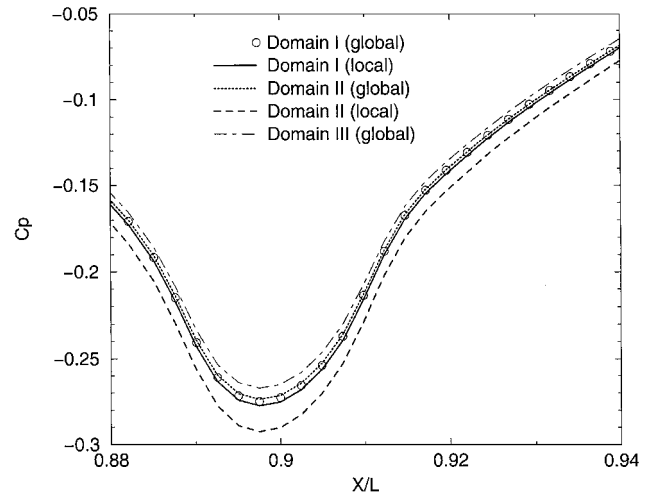


Fig. 14 Distribution of the pressure coefficient C_p on the afterbody in the symmetry plane $y = 0$: zoomed-in view.

one another very slightly. The solution obtained with global ABCs on the intermediate domain II practically does not deviate from the reference solution (domain I) as well. As opposed to global ABCs, the solution obtained with local ABCs on the intermediate domain II differs from the reference solution (domain I) much more noticeably. On the small domain III, the solution with local ABCs could not be computed at all, and the solution with global ABCs differs from the reference solution even less than the solution with local ABCs from domain II does. Similar behavior of the pressure coefficient C_p can be observed in the horizontal plane of symmetry $y = 0$ on the afterbody portion of the analyzed configuration as well (see Figs. 13 and 14).

Note that we have picked these particular C_p distributions (Figs. 11–14) as the examples of more detailed calculated flow characteristics (as opposed to presenting only integral force coefficients) because it is known that the C_p profiles on the afterbody are rather sensitive to the type and parameters of the numerical algorithm. Besides, these particular computed profiles have also been chosen by Compton in Ref. 27, and the results of our computations display a good correlation with the results by Compton that have been obtained on a very large computational domain.

All computations described in this section were conducted on Cray Research computers, J90 and C90 series. Computational overhead due to the use of global ABCs is about 15% for the particular fine grid referenced before. This overhead is determined mostly by domain geometry and typically does not scale linearly with the di-

mension of the interior grid. For the aforementioned coarse grid, the overhead reaches 30%.

Conclusions

We have constructed and implemented global ABCs for calculating external flows with jet exhaust. The ABCs combine extrapolation of all flow quantities downstream in the supersonic core of the jet and nonlocal DPM-based treatment for the remaining portion of outer boundary. The overhead associated with implementation of the new technique is compensated for by the reduced grid dimension on small domains and higher convergence rate. In the series of computations performed, the DPM-based algorithm have consistently demonstrated better accuracy, faster multigrid convergence, and higher robustness compared to the standard local methodology.

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